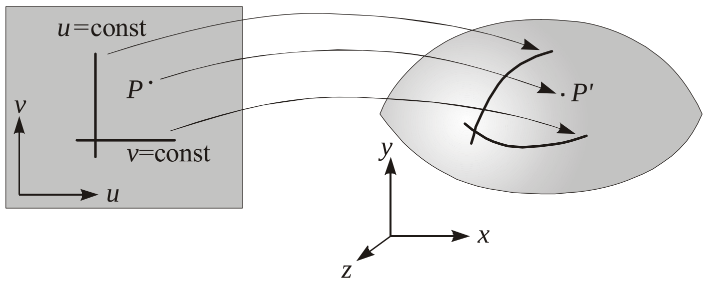


Figure 1. Defining a surface as a mapping from the *uv* plane to a surface embedded in *xyz* space. Point *P* in the *uv* plane is mapped onto point *P'* on the surface. The lines *u* = constant and *v* = constant are mapped into two curves on the surface. Taking the cross product of these two curves yields a normal to the surface. Furthermore, an element of area in the *uv* plane maps into an element of area on the surface.

Surfaces and Surface Integrals

Surfaces

We can define a surface as a mapping from the *uv* plane to a surface embedded in *xyz* space as shown in Figure 1. Write any point on the surface as $\mathbf{r} = (x, y, z)$ and write the mapping as $\mathbf{r} = \mathbf{r}(u, v)$. Then the equation

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u} du + \frac{\partial \mathbf{r}}{\partial v} dv, \tag{1}$$

describes how a small movement (du, dv) in the *uv* plane causes a small movement on the surface. In particular if we move along the line v = constant then $dv = 0$ and the movement on the surface is

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u} du,$$

i.e. in the $\frac{\partial \mathbf{r}}{\partial u}$ direction. Similarly if we move along the line u = constant then $du = 0$ and the movement on the surface is

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial v} dv,$$

i.e. in the $\frac{\partial \mathbf{r}}{\partial v}$ direction. Taking the cross-product of these two directions gives a normal **N** to the surface. Dividing **N** by its own length gives a unit normal, **n**:

$$\mathbf{N} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}, \quad \text{and} \quad \mathbf{n} = \frac{\mathbf{N}}{|\mathbf{N}|}. \tag{2}$$

Example 1: Give one possible parameterization of the plane $6x+4y+2z=8$ and use it to find the unit normal to the plane.

Solution: The plane can be parameterized as $\mathbf{r}=\mathbf{r}(u,v)=(u,\ v,\ 4-3u-2v)$. In other words we can let $x = u$, $y = v$ and then $z=4-3x-2y$ becomes $z=4-3u-2v$. Then the normal vector is

$$\mathbf{N}=\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}=(1,\ 0,-3)\times(0,\ 1,-2)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & -3\\ 0 & 1 & -2\end{vmatrix}=(3,\ 2,\ 1),$$

and the unit normal is

$$\mathbf{n}=\tfrac{1}{\sqrt{14}}(3,\ 2,\ 1).$$

Example 2: Give one possible parameterization of the sphere $x^2+y^2+z^2=a^2$ and use it to find the unit normal to the sphere.

Solution: Referring to the figure to the right, recall that spherical coordinates are defined by:

$$\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases}$$

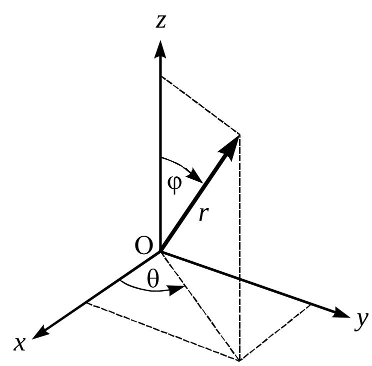


Figure 2

Thus we could parameterize the sphere as

$$\mathbf{r}=\mathbf{r}(u,v)=(a\sin u\cos v,\ a\sin u\sin v,\ a\cos u),$$

where u and v are equivalent to φ and θ respectively. The normal vector is

$$\begin{aligned}\mathbf{N}&=\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\\ &=(a\cos u\cos v,\ a\cos u\sin v,\ -a\sin u)\times(-a\sin u\sin v,\ a\sin u\cos v,\ 0)\\ &=a^2\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos u\cos v & \cos u\sin v & -\sin u\\ -\sin u\sin v & \sin u\cos v & 0\end{vmatrix}\\ &=a^2\sin u\ (\sin u\cos v,\sin u\sin v,\cos u)\end{aligned}$$

Notice that in the last expression the quantity in brackets is proportional to the vector **r** itself as expected. The unit normal is

$$\mathbf{n}=(\sin u\cos v,\sin u\sin v,\cos u),$$

as is easy to verify. Notice also that specifying u and v specifies a point **r** and a normal **n** on the sphere.

Surface Integrals

We are interested in two types of surface integrals. The first kind (the second one mentioned in Kreyszig, page 501) is of the form

$$\iint_S G(\mathbf{r})\,dA, \tag{3}$$

where *S* is the surface, *dA* is an element of area of the surface, and $G(\mathbf{r})$ is a scalar field defined at every point **r** on the surface. For example if $G(\mathbf{r})$ is the charge per unit area then this integral will yield the total charge on the surface. If $G(\mathbf{r})$=1 then this integral will simply yield the total area of the surface. This kind of surface integral does not distinguish between the back and front of the surface.

The second kind of surface integral (page 496 of Kreyszig) does. It is of the form

$$\iint_S \mathbf{F}(\mathbf{r})\cdot\mathbf{n}(\mathbf{r})\,dA. \tag{4}$$

where *S* is an orientable surface, *dA* is an element of area of the surface, **F** is a vector field defined at every point **r** on the surface and **n** is a unit vector that at every point of the surface is normal to the surface and points *out* of the surface.

This type of integral occurs for example when $\mathbf{F}=\rho\,\mathbf{v}$, where ρ is the mass density field (dimensions: mass/volume) and **v** is the velocity field (dimensions: distance/time). Then Eqn.(4) yields the mass flux (dimensions: mass/time), i.e. the mass per unit time passing through surface *S*, so (4) is often called a **flux integral**. The figure to the right shows that $\rho\,\mathbf{v}\cdot\mathbf{n}\,dA\,dt$ is the mass crossing surface element *dA* in time *dt*. Note the presence of the dot product. For example if **n** and **v** are parallel then the mass crossing the surface is a maximum and if **n** and **v** are perpendicular then the mass crossing the surface is zero.

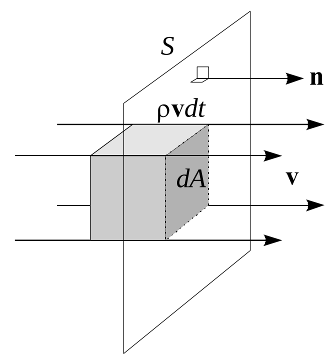


Figure 3

To actually compute one of the surface integrals (3) or (4) we use the following technique: we let *du* and *dv* be differentials in the *uv* plane, the product *du dv* be an element of area in the *uv* plane, and $\iint_R du\,dv$ be the entire area of a region *R* in the *uv* plane. The corresponding quantities generated on the surface are shown in this table:

quantity	**generated in the *uv* plane**	**generated on the surface *S***
element of length	*du* or *dv*	$\frac{\partial\mathbf{r}}{\partial u}du$ or $\frac{\partial\mathbf{r}}{\partial v}dv$
element of area	*du dv*	$\left\lvert\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right\rvert du\,dv$
total area	$\iint_R du\,dv$	$\iint_R\left\lvert\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right\rvert du\,dv$

Thus the first type of surface integral can be evaluated as

$$\iint_S G(\mathbf{r})\,dA = \iint_R G(\mathbf{r}(u,v))\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| du\,dv, \tag{5}$$

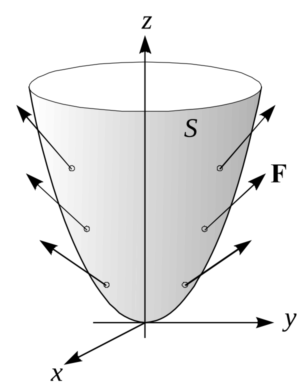
and the second type of integral can be evaluated as

$$\iint_S \mathbf{F}(\mathbf{r}) \cdot \mathbf{n}(\mathbf{r})\,dA = \iint_R \mathbf{F}(\mathbf{r}(u,v)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) du\,dv. \tag{6}$$

Note that if we let G = 1 in Eqn.(5) then we get the following **formula for computing the surface area**:

$$S = \iint_S dA = \iint_R \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| du\,dv. \tag{7}$$

Example 3: (Kreyszig, Page 503, # 9) Evaluate the flux integral $\iint_S \mathbf{F} \cdot \mathbf{n}\,dA$ in the case that the vector field is $\mathbf{F} = (x, y, z)$ and surface S is defined by $\mathbf{r} = (x, y, z) = \left(u\cos v,\, u\sin v,\, u^2\right),\ 0 \le u \le 4,\ -\pi \le v \le \pi$.



Solution: Set up Eqn. (6). The surface S and the field $\mathbf{F}$ are shown.

Step 1 - Find the field on the surface: Notice that given any point (x, y, z) in space the field $\mathbf{F}$ is determined and given any value of u and v a point on the surface is determined. By substituting the equations defining the surface, $x = u\cos v,\ y = u\sin v,\ z = u^2$, into the field $\mathbf{F}$ we find that the field at any point on the surface is given by

$$\mathbf{F}(\mathbf{r}(u,v)) = \left(u\cos v,\, u\sin v,\, u^2\right).$$

Step 2 - Find the normal $\mathbf{N} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ at any point on the surface: We find

$$\mathbf{N} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = (\cos v,\, \sin v,\, 2u) \times (-u\sin v,\, u\cos v,\, 0) = \left(-2u^2\cos v,\, -2u^2\sin v,\, u\right).$$

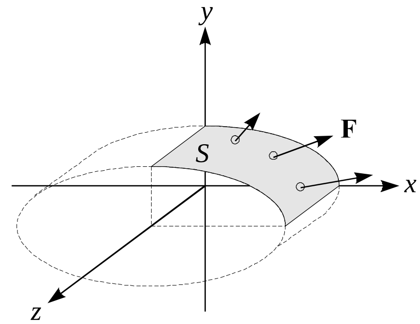
Step 3 - Evaluate the dot product of F and N on the surface:

$$\mathbf{F}(u,v) \cdot \mathbf{N}(u,v) = \left(u\cos v,\, u\sin v,\, u^2\right) \cdot \left(-2u^2\cos v,\, -2u^2\sin v,\, u\right) = -u^3.$$

Step 4 - Evaluate the surface integral by integrating over *u* and *v*:

$$\iint_R \mathbf{F}(u,v) \cdot \mathbf{N}(u,v)\,du\,dv = \int_0^4 \int_{-\pi}^{\pi} -u^3\,dv\,du = -2\pi \times \frac{4^4}{4} = -128\pi.$$

Example 4: (Kreyszig, Page 503, # 6) Evaluate the flux integral $\iint_S \mathbf{F}\cdot\mathbf{n}\,dA$ in the case that the vector field is $\mathbf{F} = \left(y^3, x^3, z^3\right)$ and surface S is defined by $x^2 + 4y^2 = 1$, $x \ge 0,\ y \ge 0,\ 0 \le z \le h$.



Solution: The surface S and the field $\mathbf{F}$ are shown to the right.

Step 0 - Express the surface in parametric form: One possibility is to let $x = u,\ y = \tfrac{1}{2}\sqrt{1-u^2},\ z = v$, with $0 \le u \le 1,\ 0 \le v \le h$.

Step 1 - Find the field on the surface: By substituting the equations defining the surface into the field $\mathbf{F}$ we find that the field at any point on the surface is given by

$$\mathbf{F}\big(\mathbf{r}(u,v)\big) = \left(\tfrac{1}{8}(1-u^2)^{3/2},\ u^3,\ v^3\right).$$

Step 2 - Find the normal $\mathbf{N} = \dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v}$ at any point on the surface: We find

$$\mathbf{N} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \left(1,\ -\frac{1}{2}\frac{u}{\sqrt{1-u^2}},\ 0\right) \times (0,\ 0,\ 1) = \left(-\frac{1}{2}\frac{u}{\sqrt{1-u^2}},\ -1,\ 0\right).$$

Step 3 - Evaluate the dot product of F and N on the surface:

$$\mathbf{F}(u,v)\cdot\mathbf{N}(u,v) = \left(\tfrac{1}{8}(1-u^2)^{3/2},\ u^3,\ v^3\right)\cdot\left(-\frac{1}{2}\frac{u}{\sqrt{1-u^2}},\ -1,\ 0\right) = -\frac{1}{16}u\left(1-u^2\right) - u^3$$

Step 4 - Finish by integrating over *u* and *v*:

$$\iint_R \mathbf{F}(u,v)\cdot\mathbf{N}(u,v)\,du\,dv = \int_0^1 \int_0^h \left(-\tfrac{1}{16}u - \tfrac{15}{16}u^3\right) dv\,du = -\tfrac{17}{64}h.$$

Example 5: Do the above example using Maple.

```
Define the field F in terms of x, y and z.
> F := [y^3,x^3,z^3];
```

$$F := \left[y^3, x^3, z^3\right]$$

```
Define the surface parameterized in terms of u and v.
> r := [u,sqrt(1-u^2)/2,v];
```

$$r := \left[u,\ \frac{1}{2}\sqrt{1-u^2},\ v\right]$$

```
Step 1 - Find the field on the surface:
> FonS := subs({x=r[1], y=r[2], z=r[3]}, [F[1],F[2],F[3]]);
```

$$FonS := \left[\frac{1}{8}\left(1-u^2\right)^{3/2}, u^3, v^3\right]$$

```
Step 2 - Find the normal N=drdu x drdv, where:
> drdu := [diff(r[1],u), diff(r[2],u), diff(r[3],u)];
```

$$drdu := \left[1, -\frac{1}{2}\frac{u}{\sqrt{1-u^2}}, 0\right]$$

```
> drdv := [diff(r[1],v), diff(r[2],v), diff(r[3],v)];
```

$$drdv := [0, 0, 1]$$

```
Load the linear algebra library.  Then we can take the crossproduct.
> with(linalg):
> N:=crossprod(drdu,drdv);
```

$$N := \begin{bmatrix} -\frac{1}{2}\frac{u}{\sqrt{1-u^2}} & -1 & 0 \end{bmatrix}$$

```
Step 3 - take the dot product of F with N
> integrand:=FonS[1]*N[1]+FonS[2]*N[2]+FonS[3]*N[3];
```

$$integrand := -\frac{1}{16}\left(1-u^2\right)u - u^3$$

```
Step 4 - do the integrals
> inner_int:=Int(integrand,v=0..h);
```

$$inner\_int := \int_0^h \left(-\frac{1}{16}\left(1-u^2\right)u - u^3\right) dv$$

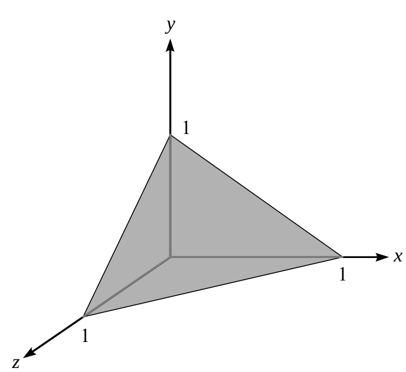
```
> surface_integral:=Int(inner_int,u=0..1);
```

$$surface\_integral := \int_0^1 \int_0^h \left(-\frac{1}{16}\left(1-u^2\right)u - u^3\right) dv\, du$$

```
> value(surface_integral);
```

$$-\frac{17}{64}h$$

Example 6: (Kreyszig, Page 504, # 12) Evaluate the surface integral $\iint_S G(\mathbf{r})\,dA$ in the case that the scalar field is $G = \cos x + \sin y$ and surface S is the portion of plane shown.



Solution:

Step 0 - Express the surface in parametric form: One possibility is to let $x = u$, $y = v$, $z = 1-u-v$, with $0 \le v \le 1-u$, $0 \le u \le 1$.

Step 1 - Find the field on the surface: By substituting the equations defining the surface into the field we find that the field at any point on the surface is given by $G = \cos u + \sin v$.

Step 2 - Find the normal $\mathbf{N} = \dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v}$ at any point on the surface: We find:

$$\mathbf{N} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = (1, 0, -1) \times (0, 1, -1) = (1, 1, 1).$$

We could have guessed this result since the plane has the equation $x + y + z = 1$. Note that $|\mathbf{N}| = \sqrt{3}$.

Step 3 - Finish by integrating over *u* and *v*:

$$\iint_S G(\mathbf{r})\,dA = \iint_R G(u,v)|\mathbf{N}(u,v)|\,du\,dv = \sqrt{3}\int_0^1 \int_0^{1-u} (\cos u + \sin v)\,dv\,du = \sqrt{3}\,(2 - \cos(1) - \sin(1)).$$

Here is the Maple dialog to evaluate the above surface integral:

```
Define the field G in terms of x, y and z
> G:=cos(x)+sin(y);
```

$$G := \cos(x) + \sin(y)$$

```
Define the surface parameterized in terms of u and v
> r:=[u,v,1-u-v];
```

$$r := [u, v, 1 - u - v]$$

```
> GonS:=subs({x=r[1],y=r[2],z=r[3]},G);
```

$$GonS := \cos(u) + \sin(v)$$

```
> drdu:=[diff(r[1],u),diff(r[2],u),diff(r[3],u)];
```

$$drdu := [1, 0, -1]$$

```
> drdv:=[diff(r[1],v),diff(r[2],v),diff(r[3],v)];
```

$$drdv := [0, 1, -1]$$

```
> N:=crossprod(drdu,drdv);
```

$$N := \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

```
> magN:=sqrt(dotprod(N,N));
```

$$magN := \sqrt{3}$$

```
> integrand:=GonS*magN;
```

$$integrand := (\cos(u) + \sin(v))\sqrt{3}$$

```
> inner_int:=Int(integrand,v=0..1-u):
> surface_integral:=Int(inner_int,u=0..1);
```

$$surface\_integral := \int_0^1 \int_0^{1-u} (\cos(u) + \sin(v))\sqrt{3}\,dv\,du$$

```
> value(surface_integral);
```

$$2\sqrt{3} - \sqrt{3}\sin(1) - \sqrt{3}\cos(1)$$

```
> evalf(%);
```

$$1.07080007$$